Lecture 21. Multiple Eigenvalue Solutions

In this section we discuss the situation when the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{1}$$

does not have n distinct roots, and thus has at least one repeated root.

An eigenvalue is of **multiplicity** k if it is a k-fold root of Eq. (1).

1. Complete Eigenvalues

- We call an eigenvalue of multiplicity k **complete** if it has k linearly independent associated eigenvectors.
- If every eigenvalue of the matrix $\mathbf{A}$ is complete, then - because eigenvectors associated with different eigenvalues are linearly independent-it follows that $\mathbf{A}$ does have a complete set of n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$(each repeated with its multiplicity).
- In this case a general solution of Eq. (1) is still given by the usual combination

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}$$

Example 1 (An example of a complete eigenvalue)

Find the general solution of the systems in the following problem.

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$

ANS: The characteristic equation of the coefficient matrix A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 9-\lambda & 7 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 (9-\lambda) = 0$$

Thus $\lambda = 2, 2, 9$.

- Case $\lambda_1 = 2$. We solve $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$.

  That is, $(A - \lambda_1 I)\vec{v} = \begin{bmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -a+b+c=0$

- If $c=0, -a+b=0$.

We can take $a=b=1$. Then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector to $\lambda_1 = 2$.

- If $b=0$, then $-a+c=0$.

  We can take $a=c=1$. Then $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is another eigenvector to $\lambda_1 = 2$.

  Note $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

- Case $\lambda_2 = 9$. We solve

$$(A-9I)\mathbf{v}_3 = \begin{bmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{cases} a=0 \\ a+c=0 \\ c=0 \end{cases}.$

Let $b=1$. Then $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_2 = 9$.

Then the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{4t}$$

2. Defective Eigenvalues

(i.e. λ has less than k linearly independent eigenvectors)

- An eigenvalue λ of multiplicity $k > 1$ is called **defective** if it is not complete.
- If the eigenvalues of the $n \times n$ matrix $\mathbf{A}$ are not all complete, then the eigenvalue method will produce fewer than the needed n linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- An example of this is the following **Example 2**.
- The defective eigenvalue $\lambda_1 = 5$ in Example 2 has multiplicity $k = 2$, but it has only 1 associated eigenvector.

The Case of Multiplicity $k = 2$

Remark: The method of finding the solutions is summarized in the **Algorithm Defective Multiplicity 2 Eigenvalues**. The following steps explain why this algorithm works.

- Let us consider the case $k = 2$, and suppose that we have found (as in Example 2) that there is only a single eigenvector $\mathbf{v}_1$ associated with the defective eigenvalue λ.
- Then at this point we have found only the single solution $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

> Recall when solving $ax'' + bx' + cx = 0$.
>
> If $ar^2 + br + c = 0$ has repeated roots. Then two linearly independent solutions are e^{rt}, te^{rt}

- By analogy with the case of a repeated characteristic root for a single linear differential equation, we might hope to find a second solution of the form

$$\mathbf{x}_2(t) = (\mathbf{v}_2 t)e^{\lambda t} = \mathbf{v}_2 t e^{\lambda t}$$

- When we substitute $\mathbf{x} = \mathbf{v}_2 t e^{\lambda t}$ in $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we get the equation

$$\mathbf{v}_2 e^{\lambda t} + \lambda \mathbf{v}_2 t e^{\lambda t} = \mathbf{A}\mathbf{v}_2 t e^{\lambda t}$$

- But because the coefficients of both $e^{\lambda t}$ and $te^{\lambda t}$ must balance, it follows that $\mathbf{v}_2 = \mathbf{0}$, and hence that $\mathbf{x}_2(t) \equiv \mathbf{0}$.
- This means that - contrary to our hope - the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ does not have a nontrivial solution of the form we assumed.
- Let us extend our idea slightly and replace $\mathbf{v}_2 t$ with $\mathbf{v}_1 t + \mathbf{v}_2$.
- Thus we explore the possibility of a second solution of the form

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero constant vectors.

- When we substitute $\mathbf{x} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ in $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we get the equation

$$\mathbf{v}_1 e^{\lambda t} + \lambda \mathbf{v}_1 t e^{\lambda t} + \lambda \mathbf{v}_2 e^{\lambda t} = \mathbf{A}\mathbf{v}_1 t e^{\lambda t} + \mathbf{A}\mathbf{v}_2 e^{\lambda t}$$

$$\Rightarrow \begin{cases} \lambda \vec{v}_1 \cancel{te^{\lambda t}} = A\vec{v}_1 \cancel{te^{\lambda t}} & \Rightarrow A\vec{v}_1 = \lambda \vec{v}_1 \\ \vec{v}_1 \cancel{e^{\lambda t}} + \lambda \vec{v}_2 \cancel{e^{\lambda t}} = A\vec{v}_2 \cancel{e^{\lambda t}} & \Rightarrow \vec{v}_1 + \lambda I \vec{v}_2 = A\vec{v}_2 \Rightarrow (A-\lambda I)\vec{v}_2 = \vec{v}_1 \end{cases}$$

- We equate coefficients of $e^{\lambda t}$ and $te^{\lambda t}$ here, and thereby obtain the two equations

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1=\mathbf{0} \quad \text{and} \quad (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2=\mathbf{v}_1$$

  that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy in order for

$$\mathbf{x}_2(t)=(\mathbf{v}_1 t+\mathbf{v}_2)e^{\lambda t}=\mathbf{v}_1 te^{\lambda t}+\mathbf{v}_2 e^{\lambda t}$$

  to give a solution of $\mathbf{x}'=\mathbf{A}\mathbf{x}$.
- Note that the first of these two equations merely confirms that $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ.
- Then the second equation says that the vector $\mathbf{v}_2$ satisfies

$$(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2=(\mathbf{A}-\lambda\mathbf{I})\left[(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2\right]=(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1=\mathbf{0}$$

- It follows that, in order to solve the two equations simultaneously, it suffices to find a solution $\mathbf{v}_2$ of the single equation $(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2=\mathbf{0}$ such that the resulting vector $\mathbf{v}_1=(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2$ is nonzero.

Algorithm Defective Multiplicity 2 Eigenvalues

1. First find nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2=\mathbf{0} \tag{2}$$

such that

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2=\mathbf{v}_1 \tag{3}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\mathbf{x}_1(t)=\mathbf{v}_1 e^{\lambda t} \tag{4}$$

and

$$\mathbf{x}_2(t)=(\mathbf{v}_1 t+\mathbf{v}_2)e^{\lambda t} \tag{5}$$

of $\mathbf{x}'=\mathbf{A}\mathbf{x}$ corresponding to λ.

Remarks: ① By the discussion in Lecture 18 we need to find $\vec{x}_1(t)$ and $\vec{x}_2(t)$ that are linearly independent.

② Note the above algorithm produces two solutions $\vec{x}_1$, $\vec{x}_2$ that are linearly independent.

③ Note $\vec{v}_1$ and $\vec{v}_2$ are not unique!

But they satisfy $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

Generalized Eigenvectors

The vector $\mathbf{v}_2$ in Eq. (2) is an example of a **generalized eigenvector**. If λ is an eigenvalue of the matrix A, then a rank r generalized eigenvector associated with λ is a vector $\mathbf{v}$ such that

$$(\mathbf{A}-\lambda\mathbf{I})^r\mathbf{v} = \mathbf{0} \quad \text{but} \quad (\mathbf{A}-\lambda\mathbf{I})^{r-1}\mathbf{v} \neq \mathbf{0}. \tag{6}$$

The vector $\mathbf{v}_2$ in (2) is a rank 2 generalized eigenvector (and not an ordinary eigenvector).

Example 2 (λ with multiplicity 2, and λ is defective)

Find the general solution of the system in the following problem. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the system.

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \mathbf{x}$$

ANS: Find the eigenvalue of A

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda) + 16 = \lambda^2 - 10\lambda + 25$$

$$= (\lambda - 5)^2 = 0$$

$\Rightarrow \lambda = 5$ with multiplicity 2.

$$(A - 5I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow a + b = 0 \Rightarrow a = -b.$$

The eigenvector corresponds to $\lambda = 5$ is a multiple of

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus λ has multiplicity 2 but only has one linearly independent eigenvector.

We apply the above algorithm to find $\vec{v}_2$ and $\vec{v}_1$.

We solve

$$\vec{0} = (A - 5I)^2 \vec{v}_2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So any a, b satisfy this eqn.

Let's choose $a=1$, $b=0$. Then $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $(A-5I)\vec{v}_2 = \vec{v}_1 \Rightarrow \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} \triangleq \vec{v}_1$

Note $\vec{v}_1$ is an eigenvector to $\lambda = 5$.

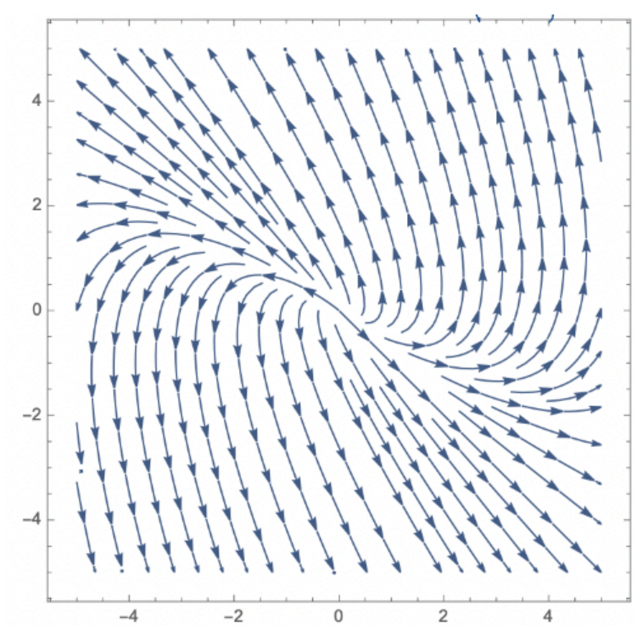
We have $\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} e^{5t}$

and

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t} = \left(\begin{bmatrix} -4t \\ 4t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^{5t}.$$

Then the general solution is

$$\vec{x}(t) = C_1 \vec{x}_1(t) + C_2 \vec{x}_2(t) = C_1 \begin{bmatrix} -4 \\ 4 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} -4t+1 \\ 4t \end{bmatrix} e^{5t}$$



Remark ④: Note $\vec{v}_1$ and $\vec{v}_2$ are not unique but related by

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1.$$

For example, given $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ we should find $\vec{v}_2$ s.t

$$(A-5I)\vec{v}_2 = \vec{v}_1 \Rightarrow \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $b=0$, then $a = \frac{1}{4}$. So $\vec{v}_2$ can be $\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$ assciated with $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Here is an online direction field calculator that generates the graph

https://www.geogebra.org/m/QPE4PaDZ

Example 3. Find the most general real-valued solution to the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \mathbf{x}$$

ANS: Find the eigenvalues of A.

$$0 = |A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix} = (\lambda+2)(\lambda+4)+1 = \lambda^2+6\lambda+9$$
$$= (\lambda+3)^2 = 0$$

$\Rightarrow \lambda = -3, -3.$

Check if $\lambda = -3$ is defective:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -2+3 & 1 \\ -1 & -4+3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a+b=0$ Any eigenvector corresponds to $\lambda = -3$ is a multiple of $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So $\lambda = -3$ is defective.

We apply the algorithm to find $\vec{v}_2$ and $\vec{v}_1$.

We solve $(A - \lambda I)^2 \vec{v}_2 = \vec{0}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We choose $a = 1$, $b = 0$ and let $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We compute

$$\vec{v}_1 = (A - \lambda I)\vec{v}_2$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So we have

$$\vec{x}_1(t) = \vec{v}_1 e^{-3t}$$

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{-3t}$$

The general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

$$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} t+1 \\ -t \end{bmatrix} e^{-3t}$$

Vector Fields

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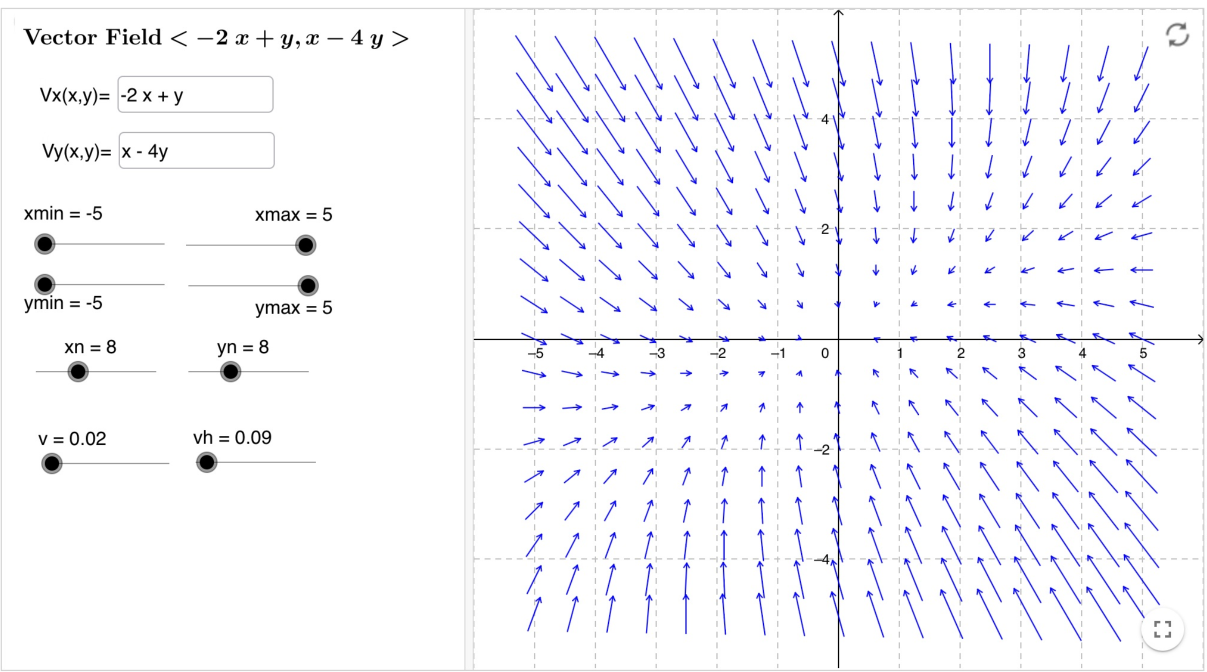
Topic: Vectors 2D (Two-Dimensional), Calculus

Change the components of the vector field.



Exercise 4. Solve the system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \mathbf{x}$$

with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Give your solution in real form.

Answer.

First, we find the eigenvalues of $A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$.

We solve

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{vmatrix} = \lambda^2 = 0$$

Thus $\lambda = 0$ with multiplicity 2. It is not hard to check that $\lambda = 0$ is defective.

So we apply the algorithm discussed in this lecture to find $\mathbf{v}_1$ and $\mathbf{v}_2$.

We first solve for $\mathbf{v}_2$ from $(A - \lambda I)^2 \mathbf{v}_2 = \mathbf{0}$.

We have

$$(A - \lambda I)^2 \mathbf{v}_2 = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{v}_2 = \mathbf{0}.$$

Thus any $\mathbf{v}_2$ would work. Let's take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then we compute $\mathbf{v}_1 = (A - \lambda I)\mathbf{v}_2 = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Therefore, we have two linearly independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{0t} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} = \left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{0t} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \left(t \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

We plug in the initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ to $\mathbf{x}(t)$, we have

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

So $3c_1 + c_2 = 2$ and $-c_1 = 4$. Therefore, $c_1 = -4$ and $c_2 = 14$.

Thus the particular solution to the initial value problem is

$$\mathbf{x}(t) = -4 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + 14 \left(t \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Remark: The general solution might vary by your choice of $\mathbf{v}_1$ and $\mathbf{v}_2$. But the particular solution to the initial value problem is unique. That is, each function in the entries of $\mathbf{x}(t)$ is a unique function to the initial value problem in this question.

Exercise 5. Suppose that the matrix A has repeated eigenvalue with the following eigenvector and generalized eigenvector:

$$\lambda = 3 \text{ with eigenvector } \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and generalized eigenvector } \boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Write the solution to the linear system $\boldsymbol{r}' = \mathrm{A}\boldsymbol{r}$ in the following forms.

(1) In eigenvalue/eigenvector form.

(2) In fundamental matrix form.

(3) As two equations.

Answer.

(1) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t \right) e^{3t}$

(2) In this case, the fundamental matrix $\Phi(t)$ has two colums consists $\mathbf{x}_1$ and $\mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two linearly independent solutions to the system. And we can write the general solution as $\mathbf{x}(t) = \Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

So we have

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & (3+t)e^{3t} \\ 2e^{3t} & (4+2t)e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(3) We need to describe the solutions to the two unknow functions explicitly, that is,

$$x(t) = c_1 e^{3t} + c_2(3+t)e^{3t}$$
$$y(t) = 2c_1 e^{3t} + c_2(4+2t)e^{3t}$$